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1/4 BPS circular loops, unstable world-sheet instantons and the matrix model

Nadav Drukker

The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen, Denmark
E-mail: drukker@nbi.dk

Abstract: The standard prescription for computing Wilson loops in the AdS/CFT correspondence in the large coupling regime and tree-level involves minimizing the string action. In many cases the action has more than one saddle point as in the simple example studied in this paper, where there are two 1/4 BPS string solutions, one a minimum and the other not. Like in the case of the regular circular loop the perturbative expansion seems to be captured by a free matrix model. This gives enough analytic control to extrapolate from weak to strong coupling and find both saddle points in the asymptotic expansion of the matrix model. The calculation also suggests a new BMN-like limit for nearly BPS Wilson loop operators.

Keywords: AdS-CFT Correspondence, Matrix Models.

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Contents

1. Introduction

This note deals with a family of circular Wilson loop operators in $\mathcal{N}=4$ supersymmetric Yang-Mills theory which have certain couplings to three of the six scalar fields. They may be written as

$$W_{\theta_0}=\frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\left[\int\left(iA_\alpha\dot{x}^\alpha(\tau)+|\dot{x}(\tau)|\Theta^I(\tau)\Phi_I\right)d\tau\right], \tag{1.1}$$

where $0<\tau<2\pi$, the path in space $x^\alpha(\tau)$ is a circle of radius R

$$x^1=R\cos\tau\,,\qquad x^2=R\sin\tau\,, \tag{1.2}$$

and they couple to the three scalars Φ_1, Φ_2 and Φ_3 in the following way

$$\Theta^1=\sin\theta_0\cos\tau\,,\qquad\Theta^2=\sin\theta_0\sin\tau\,,\qquad\Theta^3=\cos\theta_0\,, \tag{1.3}$$

with an arbitrary fixed θ_0.

These operators were first presented in [1] where they were evaluated using the AdS/CFT correspondence [2 − 4] by a classical string surface. But these operators are quite interesting and deserves more attention. As we shall see they preserve 1/4 of the supersymmetries of the vacuum and by varying θ_0 it is possible to interpolate from the usual circle [5, 6] (at $\theta_0=0$) that couples to only one of the scalars and preserves half of the supersymmetries, to the 1/4 BPS circular loop [7] for $\theta_0=\pi/2$ which couples to two scalars.

Evaluating these Wilson loops in AdS requires finding a surface which partially wraps an $S^2\subset S^5$. There are two ways to do that, over the northern and southern poles, and the resulting values of the classical action are $\pm\sqrt{\lambda}\cos\theta_0$ where $\lambda=g^2_{\mathrm{YM}}N$ is the 't Hooft coupling. On the gauge theory side evaluating this operator in perturbation theory is very

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similar to the usual circle at $\theta_0 = 0$ and at least to 2-loop order is captured by a Gaussian matrix model [8, 9]. The only modification is the replacement of $\lambda \to \lambda' = \lambda \cos^2\theta_0$, hence the standard agreement of the string calculation with the matrix model result carries over.

What is very intriguing about the AdS calculation is that there is more than one saddle point. In general one expects the string expansion to be asymptotic, and indeed an extra exponentially suppressed saddle point is found contributing $\exp -\sqrt{\lambda'}$ to the expectation value of the Wilson loop. Up to now only the dominant saddle point was considered and was indeed found also from the matrix model. But at the planar level the matrix model is given by a Bessel function whose asymptotic expansion at large argument has two saddle points, which matches the AdS calculation including the subleading one! The rest of the paper contains the details of this remarkable agreement.

2. Gauge theory calculation

Let us start by calculating those circular Wilson loop observables in perturbation theory. At order $g^2_{\rm YM}$ one should sum over the gauge field and scalar exchange

$$\begin{aligned}\langle W\rangle = 1 + \frac{1}{2N}\int d\tau_1\, d\tau_2\, \mathrm{Tr}\, T^a T^b \Big[& -\dot{x}^\alpha(\tau_1)\dot{x}^\beta(\tau_2) G^{ab}_{\alpha\beta}(x(\tau_1), x(\tau_2)) \\ & + |\dot{x}^\alpha(\tau_1)||\dot{x}^\beta(\tau_2)|\Theta^I(\tau_1)\Theta^I(\tau_2) G^{ab}(x(\tau_1), x(\tau_2))\Big]\,.\end{aligned} \tag{2.1}$$

Here T^a are generators of the gauge group and they satisfy $T^aT^a = N/2\times I$, where I is the identity matrix. $G^{ab}_{\alpha\beta}$ is the gauge propagator and G^{ab} the scalar propagator, and in the Feynman gauge they are given by

$$G^{ab}_{\alpha\beta}(x_1, x_2) = \frac{g^2_{\rm YM}\delta_{\alpha\beta}\delta^{ab}}{(x_1 - x_2)^2}\,, \qquad G^{ab}(x_1, x_2) = \frac{g^2_{\rm YM}\delta^{ab}}{(x_1 - x_2)^2}\,. \tag{2.2}$$

Therefore at order $g^2_{\rm YM}$ the circle is given by

$$\langle W\rangle = 1 + \frac{g^2_{\rm YM}N}{(4\pi)^2}\int d\tau_1\, d\tau_2 \frac{-\cos(\tau_1-\tau_2) + \sin^2\theta_0\cos(\tau_1-\tau_2) + \cos^2\theta_0}{4\sin^2(\tau_1-\tau_2)/2}\,. \tag{2.3}$$

Remarkably, the integrand is a constant, $\frac{1}{2}\cos^2\theta_0$, as in the case studied in [8], which is recovered by taking $\theta_0 = 0$. Thus one finds

$$\langle W\rangle = 1 + \frac{g^2_{\rm YM}N}{8}\cos^2\theta_0 + O(g^4)\,. \tag{2.4}$$

In the case where $\theta_0 = 0$ it was shown [8] that the interacting graphs in the Feynman gauge at order g^4 vanish. The same also happens at $\theta_0 = \pi/2$, where the expectation value of the Wilson loop is unity. Quite remarkably this extends to arbitrary θ_0. It is possible to separate the graphs to contributions associated to Φ_3 and those associated to Φ_1 and Φ_2. The graphs involving Φ_3 will be identical (up to a constant) to those for $\theta_0 = 0$ and vanish by the same calculation of [8]. The graphs involving the other two scalars are the same as for $\theta_0 = \pi/2$ and will vanish as well.

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So the contribution at order $g_{\rm YM}^4$ comes only from non-interacting graphs, where the propagator is a constant. It is then reasonable to conjecture that at higher orders interacting graphs still don't contribute and the full result will be given by the sum of ladders, as in the case of $\theta_0 = 0$ [8, 9].

Under that assumption the Wilson loop will be given by the sum of all non-interacting diagrams which is easily written in terms of a 0-dimensional Hermitian Gaussian matrix model. The θ_0 dependence will show up in the coupling constant, where replacing $\lambda \to \lambda' = \lambda \cos^2\theta_0$ will give the correct normalization of the matrix propagator

$$\langle W_{\theta_0}\rangle_{\rm ladder} = \left\langle \frac{1}{N}{\rm Tr}\,\exp(M)\right\rangle_{\rm 0d} = \frac{1}{Z}\int \mathcal{D}M\,\frac{1}{N}{\rm Tr}\,\exp(M)\exp\left(-\frac{2N}{\lambda'}{\rm Tr}\,M^2\right). \tag{2.5}$$

The full large N expansion of this matrix model was given in [9]. The leading result at large λ is [8]

$$\langle W_{\theta_0}\rangle_{\rm ladder} \sim \exp\sqrt{\lambda}\,|\cos\theta_0| = \exp\sqrt{\lambda'}\,. \tag{2.6}$$

2.1 Supersymmetry

The vacuum of $\mathcal{N}=4$ super Yang-Mills has 32 supersymmetries which are generated by the spinors

$$\epsilon(x) = \epsilon_0 + \sigma_\alpha x^\alpha \epsilon_1\,, \tag{2.7}$$

with constant ϵ_0 and ϵ_1. Here σ_α are the usual Dirac matrices and the equations below will also involve ρ^i, which act on the SO(6) indices of ϵ. The conventions used are those of dimensionally reduced $\mathcal{N}=1$ super Yang-Mills in ten dimensions, so all the gamma matrices σ^α and ρ^i anti-commute.

At the linear order the supersymmetry variation of the Wilson loop is proportional to

$$\Big[-i\sigma_1\sin\tau + i\sigma_2\cos\tau + \sin\theta_0(\rho_1\cos\tau+\rho_2\sin\tau) + \rho_3\cos\theta_0\Big]\epsilon(x)\,, \tag{2.8}$$

A Wilson loop will be supersymmetric if the above expression vanishes for some of the components of $\epsilon(x)$, which in this case is

$$\epsilon(x) = \epsilon_0 + R(\sigma_1\cos\tau + \sigma_2\sin\tau)\epsilon_1\,. \tag{2.9}$$

The resulting equation may be separated into terms with different functional dependence on τ which should all vanish independently

$$\begin{array}{lrl}
\cos\tau: & (i\sigma_2+\sin\theta_0\,\rho_1)\epsilon_0 &= -R\cos\theta_0\,\rho_3\sigma_1\epsilon_1\,,\\
\sin\tau: & (-i\sigma_1+\sin\theta_0\,\rho_2)\epsilon_0 &= -R\cos\theta_0\,\rho_3\sigma_2\epsilon_1\,,\\
\cos\tau\sin\tau: & 0 &= R\sin\theta_0(\rho_2\sigma_1+\rho_1\sigma_2)\epsilon_1\,,\\
\cos^2\tau: & 0 &= R\sin\theta_0(\rho_1\sigma_1-\rho_2\sigma_2)\epsilon_1\,,\\
1: & \cos\theta_0\,\rho_3\epsilon_0 &= R(i\sigma_1-\sin\theta_0\,\rho_2)\sigma_2\epsilon_1\,.
\end{array} \tag{2.10}$$

These conditions are not independent. First for $\sin\theta_0 = 0$ they are all solved as long as ϵ_0 and ϵ_1 are related by

$$\epsilon_0 = iR\rho_3\sigma_1\sigma_2\epsilon_1\,. \tag{2.11}$$

This configurations preserves 1/2 the supersymmetries, but they all involve some of the super-conformal transformations.

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For $\sin\theta_0 = 1$ the two constant spinors are not related, rather there are two independent conditions on each

$$(i\sigma_2 + \rho_1)\epsilon_a = (i\sigma_1 - \rho_2)\epsilon_a = 0\,. \tag{2.12}$$

Thus this solutions preserves 1/4 of the regular supersymmetries and 1/4 of the superconformal ones.

For generic θ_0 it is easy to see that all the above equations are satisfied as long as the following two relations hold

$$\begin{aligned}\cos\theta_0\epsilon_0 &= R(-i\sigma_1 + \sin\theta_0\,\rho_2)\rho_3\sigma_2\epsilon_1\,,\\ (\rho_2\sigma_1 + \rho_1\sigma_2)\epsilon_1 &= 0\,.\end{aligned} \tag{2.13}$$

Note that as a consequence the last equation holds also for ϵ_0. The Wilson loop that is studied here preserves, therefore 1/4 of the supersymmetries.

3. String theory calculation

The description of those Wilson loops by strings in $AdS_5 \times S^5$ was presented in [1] (section 4.3.2), the following is a short review (in a different coordinate system). Use the target space metric

$$\begin{aligned}ds^2 =&\ L^2\left(-\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho(d\chi^2 + \cos^2\chi\,d\psi^2 + \sin^2\chi\,d\varphi_1^2\right)\\ &+L^2\left(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta\left(d\vartheta_1^2 + \sin^2\vartheta_1\left(d\vartheta_2^2 + \sin^2\vartheta_2 d\varphi_2^2\right)\right)\right)\,.\end{aligned} \tag{3.1}$$

This is global Lorentzian AdS_5 with curvature radius L related to the 't Hooft coupling by $L^4 = \lambda\alpha'^2$. The circle will follow the coordinate ψ on the equator of the S^3 on the boundary of AdS_5. On the S^5 side the string will be inside an S^2 given by θ and ϕ (after doubling the range of θ to $[0,\pi]$ at the expense of ϑ_1). Using the ansatz

$$\rho = \rho(\sigma)\,,\qquad \psi(\tau) = \tau\,,\qquad \theta = \theta(\sigma)\,,\qquad \phi(\tau) = \tau\,,\qquad t = \chi = \vartheta_1 = 0\,, \tag{3.2}$$

leads to the action in conformal gauge

$$\mathcal{S} = \frac{L^2}{4\pi\alpha'}\int d\sigma\,d\tau\left[\rho'^2 + \sinh^2\rho + \theta'^2 + \sin^2\theta\right]\,. \tag{3.3}$$

The equations of motion are

$$\begin{aligned}\rho'' &= \sinh\rho\cosh\rho\,,\\ \theta'' &= \sin\theta\cos\theta\,,\end{aligned} \tag{3.4}$$

and the Virasoro constraint reads

$$\rho'^2 + \theta'^2 = \sinh^2\rho + \sin^2\theta\,. \tag{3.5}$$

The first integral for ρ is

$$\rho'^2 - \sinh^2\rho = c\,, \tag{3.6}$$

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and to get a surface that corresponds to a single circle and not the correlator or two one has to set $c = 0$, so the solution is

$$\sinh\rho(\sigma) = \frac{1}{\sinh\sigma}\,. \tag{3.7}$$

An integration constant in this equation that shifts σ was set to zero so the boundary of the world-sheet at $\sigma = 0$ is at the boundary of AdS_5. Then the first integral for θ is

$$\theta'^2 = \sin^2\theta\,, \tag{3.8}$$

which is solved by

$$\sin\theta(\sigma) = \frac{1}{\cosh(\sigma_0 \pm \sigma)}\,, \qquad \cos\theta(\sigma) = \tanh(\sigma_0 \pm \sigma)\,. \tag{3.9}$$

In this equation the integration constant σ_0 is important, it is fixed by the boundary condition that at $\sigma = 0$

$$\cos\theta_0 = \tanh\sigma_0\,. \tag{3.10}$$

Depending on the sign in (3.9) the surface extends over the north or south pole of S^5.

The bulk part of the classical action is proportional to the area

$$\begin{aligned} S_{\text{bulk}} &= \sqrt{\lambda}\int d\sigma\left(\sinh^2\rho + \sin^2\theta\right) = \int d\sigma\left(\frac{1}{\sinh^2\sigma} + \frac{1}{\cosh^2(\sigma_0 \pm \sigma)}\right) \\ &= \sqrt{\lambda}\left(\coth\sigma_{\min} \mp \tanh\sigma_0\right) = \sqrt{\lambda}\left(\cosh\rho_{\max} \mp \cos\theta_0\right)\,. \end{aligned} \tag{3.11}$$

Here $\sigma_{\min}$ is a cutoff on σ and $\rho_{\max}$ the corresponding cutoff on ρ. This divergent part is canceled by an extra boundary term in the action [6], so the final result is

$$\mathcal{S} = \mp\cos\theta_0\sqrt{\lambda}\,. \tag{3.12}$$

The two signs correspond to a string extended over the north or south poles of S^2. From this one finds that the expectation value of the Wilson loop at strong coupling is given by

$$\langle W\rangle \sim \exp\left[\pm\cos\theta_0\sqrt{\lambda}\right]\,, \tag{3.13}$$

and the sign should be chosen to minimize the action.

3.1 Supersymmetry

In order to check supersymmetry choose the vielbeins (only for the directions that are turned on)

$$e^1 = L\,d\rho\,, \qquad e^3 = L\sinh\rho\,d\psi\,, \qquad e^5 = L\,d\theta\,, \qquad e^6 = L\sin\theta\,d\phi\,. \tag{3.14}$$

Γ_a will be ten real constant gamma matrices and define $\gamma_\mu = e^a_\mu\Gamma_a$ and $\Gamma_\star = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4$ the product of all the gamma matrices in the AdS_5 directions. With this the dependence of the Killing spinors on the relevant coordinates may be written as (see for example [10 – 13])

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{\frac{1}{2}\psi\,\Gamma_{13}}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\phi\,\Gamma_{56}}\epsilon_0\,, \tag{3.15}$$

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where ϵ_0 is a chiral complex 16-component spinor. This satisfies the Killing spinor equation[1]

$$\left(D_\mu + \frac{i}{2L}\Gamma_\star\gamma_\mu\right)\epsilon = 0\,. \tag{3.16}$$

The projector associated with a fundamental string in type IIB is

$$\Gamma = \frac{1}{\sqrt{g}}\,\partial_\tau x^\mu \partial_\sigma x^\nu \gamma_\mu\gamma_\nu K\,, \tag{3.17}$$

where g is the induced metric on the world-sheet and K acts on spinors by complex conjugation. The number of supersymmetries preserved by the string is the number of independent solutions to the equation $\Gamma\epsilon = \epsilon$.

For the two solutions (the signs correspond to the two choices in (3.9))

$$\Gamma = \frac{1}{\sinh^2\rho + \sin^2\theta}\left(\sinh^2\rho\,\Gamma_{13} \pm \sin^2\theta\,\Gamma_{56} + \sinh\rho\sin\theta\,\Gamma_{16} \pm \sinh\rho\sin\theta\,\Gamma_{53}\right)K\,. \tag{3.18}$$

The equation has to hold for all σ and τ. Since Γ_{13} commutes with $\Gamma_\star\Gamma_5$ and with Γ_{56} and also $\Gamma_\star\Gamma_1$ commutes with $\Gamma_\star\Gamma_5$ one may write the Killing spinor as

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1 - \frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\, e^{\frac{1}{2}\tau(\Gamma_{13}+\Gamma_{56})}\epsilon_0\,. \tag{3.19}$$

Note that Γ does not depend on τ, the only place τ appears in the projector equation is in the second exponential of this expression for the Killing spinors. To eliminate this dependence impose the condition

$$(\Gamma_{13} + \Gamma_{56})\epsilon_0 = 0\,. \tag{3.20}$$

Now commuting the terms in the projector Γ through the remaining exponential in (3.19), remembering that K acts by complex conjugation, one gets

$$\begin{aligned}\Gamma\epsilon = \frac{1}{\sinh^2\rho + \sin^2\theta}\Big[&e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1 + \frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\left(\sinh^2\rho\,\Gamma_{13} \pm \sinh\rho\sin\theta\,\Gamma_{53}\right)\\ &+ e^{\frac{i}{2}\rho\,\Gamma_\star\Gamma_1 - \frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\left(\pm\sin^2\theta\,\Gamma_{56} + \sinh\rho\sin\theta\,\Gamma_{16}\right)\Big]K\epsilon_0\,.\end{aligned} \tag{3.21}$$

Next note that by factoring out by the remaining exponential in the Killing spinor, the projector equation $\Gamma\epsilon = \epsilon$ reduces to an equation on the constant spinor $\bar\Gamma\epsilon_0 = \epsilon_0$ with

$$\begin{aligned}\bar\Gamma = \frac{1}{\sinh^2\rho + \sin^2\theta}\Big[&e^{i\theta\,\Gamma_\star\Gamma_5}\left(\sinh^2\rho\,\Gamma_{13} \pm \sinh\rho\sin\theta\,\Gamma_{53}\right)\\ &+ e^{i\rho\,\Gamma_\star\Gamma_1}\left(\pm\sin^2\theta\,\Gamma_{56} + \sinh\rho\sin\theta\,\Gamma_{16}\right)\Big]K\end{aligned} \tag{3.22}$$

By expanding the exponentials and using (3.20) the projector equation becomes

$$\begin{aligned}\bar\Gamma\epsilon_0 = \frac{1}{\sinh^2\rho + \sin^2\theta}\Big[&(\cos\theta\sinh^2\rho \mp \cosh\rho\sin^2\theta)\,\Gamma_{13}\\ &+ (\cos\theta \pm \cosh\rho)\sin\theta\sinh\rho\,\Gamma_{16}\Big]K\epsilon_0\,.\end{aligned} \tag{3.23}$$

[1] $D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}$ and the only relevant non-zero components of the spin-connection are $\omega_\psi^{13} = -\cosh\rho$ and $\omega_\phi^{56} = -\cos\theta$.

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Replacing the solutions for $\rho(\sigma)$ (3.7) and $\theta(\sigma)$ (3.9)

$$\bar{\Gamma}\epsilon_0 = \frac{1}{\cosh\sigma_0}\Big[\sinh\sigma_0\,\Gamma_{13} + \Gamma_{16}\Big]K\epsilon_0 = \Big[\cos\theta_0\,\Gamma_{13} + \sin\theta_0\Gamma_{16}\Big]K\epsilon_0\,. \tag{3.24}$$

Half the eigenvalues of this matrix (for generic σ_0) are one and it commutes with Γ_{1356}, so the two projections are compatible and the string solution preserves 1/4 of the supersymmetries.

Note that the final expression does not depend on whether the surface extends over the north or south pole of the S^2 and depends only on θ_0. Therefore both those surfaces preserve the same supersymmetries which are the same as those found on the gauge theory side.

4. Discussion

The family of Wilson loop operators considered in this note preserve 1/4 of the supersymmetries of the vacuum and allow to do some very interesting calculations. On the gauge theory side it was easy to calculate them to order $g_{\rm YM}^4$ and the result is the same as for the 1/2-BPS loop [8] with the replacement $\lambda \to \lambda' = \lambda\cos^2\theta_0$. It is therefore natural to conjecture that the final result is given by the same matrix model as in [9] with this replacement.

On the string theory side two classical string solutions describing this Wilson loop were found and both preserved the same supersymmetry. For generic θ_0 the action of those two surfaces is not equal, rather they have the opposite signs. It is quite common to find more than one saddle point to the string equations of motion [1] and there are indeed more solutions here one gets by adding extra wrappings of the sphere, but those would not be supersymmetric.

These non-supersymmetric saddle points of the string action will not contribute to the same expectation values as the supersymmetric ones due to having extra fermionic zero modes. So if the two surfaces found here are indeed the only supersymmetric world-sheets satisfying the correct boundary conditions, the Wilson loop expectation value will have a semiclassical expansion at large λ as

$$\langle W_{\theta_0}\rangle \sim e^{\sqrt{\lambda'}} + e^{-\sqrt{\lambda'}}\,. \tag{4.1}$$

Recall that at the planar level[2] the Gaussian matrix model is given by Wigner's semicircular distribution

$$\langle W_{\theta_0}\rangle_{\rm ladder} = \frac{2}{\pi\lambda'}\int_{-\sqrt{\lambda'}}^{\sqrt{\lambda'}} dx\sqrt{\lambda' - x^2}\,e^x = \frac{2}{\sqrt{\lambda'}}I_1\left(\sqrt{\lambda'}\right)\,, \tag{4.2}$$

where I_1 is the modified Bessel function. Using the asymptotic expansion of the Bessel function at large λ' one finds the result that should be reproduced by semiclassical super-

[2]For some results beyond the planar level see [14, 15]

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gravity

$$\langle W_{\theta_0}\rangle_{\text{ladder}} = \sqrt{\frac{2}{\pi}}\frac{e^{\sqrt{\lambda'}}}{\lambda'^{3/4}}\sum_{k=0}^{\infty}\left(\frac{-1}{2\sqrt{\lambda'}}\right)^k\frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2}-k)} - i\sqrt{\frac{2}{\pi}}\frac{e^{-\sqrt{\lambda'}}}{\lambda'^{3/4}}\sum_{k=0}^{\infty}\left(\frac{1}{2\sqrt{\lambda'}}\right)^k\frac{\Gamma(\frac{3}{2}+k)}{\Gamma(\frac{3}{2}-k)}\,. \tag{4.3}$$

This expression represents two saddle points with classical action $\pm\sqrt{\lambda'}$, exactly as was found here from string theory[3]. The asymptotic expansion includes an infinite series of perturbative corrections in inverse powers of $\sqrt{\lambda'}$ which should be found by doing the world sheet perturbation expansion around those solutions as was pursued in [16].

Note that on S^5 there are three transverse directions to the S^2. Around the minimum of the action those directions are massive, but not around the other saddle point. Turning on those deformations of the surface will cause it to "slip" away from that pole. Those three modes are tachyonic and each contributes a factor of i to the fluctuation determinant. This too is matched by the results of the asymptotic expansion, where the second term in (4.3) is imaginary.

Clearly the term with negative exponent will never dominate the action and its contribution is smaller than any of the $(\lambda')^{-k/2}$ corrections to the leading term. It's quite miraculous that it was possible to fit this term between the perturbative gauge theory calculation and string theory. Such results are often associated with localization theorems, which may be the case here due to the large number of supersymmetries preserved by this Wilson loop.

One can go much further than the semiclassical string calculation. It is possible to take λ large, so the string theory is still on a low curvature background while keeping $\lambda' = \cos^2\theta_0\lambda$ small, in a fashion similar to the BMN limit [17]. For $\cos\theta_0 = 0$ Zarembo's solution [7] has three zero modes parameterizing an S^3 with measure

$$d\Omega_3 = \frac{1}{2\pi^2}\,d\alpha\sin^2\alpha\,d\Omega_2\,, \tag{4.4}$$

where the range of α is $[0,\pi]$ and Ω_2 is the measure on an S^2 that remains unbroken for nonzero $\cos\theta_0$. Turning on $\cos\theta_0$ leads to a potential $\cos\alpha\cos\theta_0\sqrt{\lambda}$, so the integration over the broken zero modes gives for the Wilson loop

$$\langle W_{\theta_0}\rangle = \frac{2}{\pi}\int_0^\pi d\alpha\,\sin^2\alpha\,e^{-\cos\alpha\sqrt{\lambda'}}\,. \tag{4.5}$$

This is exactly equal to the result of the matrix model at the planar level (4.2). Here it is reproduced from perturbative string theory by the inclusion only of the zero modes. It would be interesting to see if this kind of BMN-like limit generalizes to other deformations of supersymmetric Wilson loops.

Recently the AdS description of the supersymmetric Wilson loops of the type constructed by Zarembo [7] was found [18]. Those operators preserve the regular Poincaré supersymmetries and have trivial expectation values [19, 20], but there should be many

[3]Those saddle points come from the semiclassical evaluation of the integral in (4.2) around the two endpoints.

more supersymmetric Wilson loops which preserve other combinations of the regular and conformal supersymmetry generators. Those include the ones studied in this paper as well as some deformations of the line or circle by insertion of local operators as in [21]. There is a very rich structure of supersymmetric Wilson loops that is worth exploring.

Clearly some of the usual intuitions about supersymmetry does not apply to those combinations of regular and conformal supersymmetries. Here there is an unstable surface that preserves some supersymmetry.

In the case of the 1/2 BPS circle with $\theta_0 = 0$ it is possible to map it to the line which preserves a regular supersymmetry and whose expectation value is trivial. For other values of θ_0 the resulting line operator will not be trivial, rather it will be given by the path

$$x^1 = \tau\,, \tag{4.6}$$

for $\tau \in \mathbb{R}$ and the coupling to the scalars is given by[4]

$$\Theta^1 = \sin\theta_0 \frac{\tau^2-1}{\tau^2+1}\,, \qquad \Theta^2 = \sin\theta_0 \frac{2\tau}{\tau^2+1}\,, \qquad \Theta^3 = \cos\theta_0\,. \tag{4.7}$$

It is possible to study this operator both in AdS and at weak coupling. By doing a conformal map on the AdS solution one finds a new surface with action

$$\mathcal{S} = \sqrt{\lambda}(1 \mp \cos\theta_0)\,. \tag{4.8}$$

The difference between the circle (3.12) and this case is $\sqrt{\lambda}$ in agreement with a general argument [9] that whenever a conformal transformation maps a compact Wilson loop to a non-compact one the ratio of the Wilson loop VEVs is universal. This ratio is fixed to $\exp\sqrt{\lambda}$ (or more generally to the result of the matrix model) by considering the example of $\theta_0 = 0$.

On the gauge theory side at one-loop the expectation value of the line is

$$\langle W \rangle = 1 - \frac{g_{\rm YM}^2 N}{8} \sin^2\theta_0\,, \tag{4.9}$$

which also agrees with this general argument. Note that unless $\theta_0 = 0$, the supersymmetries preserved by this line all involve the superconformal generators, and it is impossible to find a map that will turn them into operators annihilated purely by combinations of the Poincaré supersymmetry generators.

Another recent development in the study of Wilson loops is the renewed interest in the description of them in terms of D-branes [3, 14, 15, 23 – 25]. It would be very interesting to find the relevant solutions for the 1/4 BPS Wilson loops, see if there are two solutions also for the DBI action, and study the $1/N$ corrections to the two saddle points.

[4]This is different from the periodic couplings along the line considered in [22, 1], which were not supersymmetric

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